13 Vector Functions

13.1 Vector functions ad Space Curves

1. a vector function is a function whose input is a real number, and output is a vector
2. the limit of the vector function is defined by taking the limit of its component functions
   $\lim_{t\to a} \mathbf{r(t)} = \langle \lim_{\mathbf{t\to a}} \mathbf{f(t)}, \lim_{\mathbf{t\to a}} \mathbf{g(t)}, \lim_{\mathbf{t\to ah(t)}} \rangle$
3. a vector is continuous at a if $\lim_{t\to a} \mathbf{r(t)} = \lim_{\mathbf{t\to a}} \mathbf{r(a)}$
4. a vector is continuous at a iff its component functions are continuous at a.
5. the space curve is the set of all points in space (x, y, z), where $x = f(t)$, $y = g(t)$ and $z = h(t)$ (the curve traced out by the tip of the position vector of the point (x, y, z)). These equations are the parametric equations of the space curve C, and t is the parameter.
6. you may skip the computer generated curves

13.2 Derivative and Integrals of vector functions

1. the derivative is defined by $\frac{d\mathbf{r}}{dt} = \mathbf{r'(t)} = \lim_{\mathbf{t\to a}} \frac{\mathbf{r(t+h) - r(t)}}{\mathbf{h}}$
2. the derivative can be found by taking the derivative of its component functions
   $$\frac{d\mathbf{r}}{dt} = \langle f'(t), g'(t), h'(t) \rangle$$
3. $r'(t)$ is the tangent vector to the curve, and the unit tangent vector $\mathbf{T}(t) = \frac{r'(t)}{|r'(t)|}$
4. the tangent vector to C at P is the vector parallel to $r'(t)$ that goes through P
5. the second derivative $r''{}'(t) = (r'(t))'$ and see differentiation rules – page 859
6. a curve $r(t)$ is smooth if $r'(t)$ exists and it is nonzero at every point. If $r'(t)$ does not exist or it is zero at finitely many points, then the curve is piecewise smooth
7. the definite integral is defined by
   $$\int_a^b \mathbf{r(t)dt} = \Big(\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{f(t)dt}\Big)\mathbf{i} + \Big(\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{g(t)dt}\Big)\mathbf{j} + \Big(\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{h(t)dt}\Big)\mathbf{k}$$
8. Fundamental Theorem of Calc for vectors: $\int_a^b r(t)dt = \mathbf{R(b)} - \mathbf{R(a)}$, where $R(t)$ is the antiderivative of $r(t)$.

13.3 Arc length and curvature

1. the length of a space curve is similar to the length of a plane curve:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \text{ or } L = \int_a^b \Big|r'(t)\Big| dt$$

2. the arc length found using the above formula is independent of the parametrization

3. the arc length function $s(t) = \displaystyle\int_a^t |r'(u)| du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2} du.$

   Note that the answer is a function of t since the arc lenght function $s(t)$ is a function of t as well, and it gives the length of the curve from point a to point t, for all $t \leq b$. The formula in item 1 above is a number, which is the length of the curve from point a to point b.

4. parameterizing a curve with respect to arc length helps in finding the position vector s units along the curve. This parametrization does not depend on the particular coordinate system used.

5. in parameterizing a curve with respect to arc length, we use the derivative

   $\dfrac{ds}{dt} = |r'(t)|$ in finding

   $$s(t) = \int_0^t \frac{ds}{du} du = \int_0^t |r'(u)| du$$

CURVATURE

1. a curve is smooth if the derivative exists and it is continuous everywhere, and it is nonzero

2. the unit tangent vector is $\mathbf{T}(t) = \frac{\mathbf{r'(t)}}{|\mathbf{r'(t)}|}$ and it indicates the direction of the curve

3. the curvature of a curve C at a point is a measure of how quickly the curve

   changes direction at that point $\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{\frac{d\mathbf{T}}{dt}}{\frac{ds}{dt}}\right| = \frac{|\mathbf{T'(t)}|}{|\mathbf{r'(t)}|}$

4. Thm: the curvature given by r is $\kappa(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$

5. For a plane curve with equation $y = f(x)$, we can write $r(x) = xi + f(x)j$, and

   so $\kappa(x) = \frac{|f''(x)|}{[1+(f'(t))^2]^{3/2}}$

THE TANGENT, NORMAL AND BINORMAL VECTORS

1. recall: the unit tangent vector $\mathbf{T}(\mathbf{t})$ is $\mathbf{T}(t) = \frac{\mathbf{r}'(\mathbf{t})}{|\mathbf{r}'(\mathbf{t})|}$ and it indicates the direction of the curve

2. the unit normal vector $\mathbf{N}(\mathbf{t})$ is defined as $\mathbf{N}(\mathbf{t}) = \dfrac{\mathbf{T}'(\mathbf{t})}{|\mathbf{T}'(\mathbf{t})|}$ and it is always normal to the curve (i.e. orthogonal to the unit tangent vector $\mathbf{T}(\mathbf{t})$)

3. the binormal vector $\mathbf{B}(\mathbf{t})$ is $\mathbf{B}(\mathbf{t}) = \mathbf{T}(t) \times \mathbf{N}(t)$, and it is orthogonal to both $\mathbf{T}(t)$ and $\mathbf{N}(t)$ as their cross product.

4. the normal plane is the plane determined by both $\mathbf{N}(t)$ and $\mathbf{B}(t)$ at a point P (i.e. it consists of all the lines orthogonal to $\mathbf{T}(t)$)

5. the osculating plane is the plane determined by both $\mathbf{T}(t)$ and $\mathbf{N}(t)$ (for a plane curve, it is the plane that contains the curve, the normal and its tangent)

13.4 Motion in Space: Velocity and Acceleration

1. the applications of tangent and normal vectors, and curvatures to physics in studying an object's velocity and acceleration along a space curve.

2. object moves in space:
   - its *position vector* at time t is $r(t)$
   - its *direction* is approximated by the average velocity $\frac{\mathbf{r}(\mathbf{t}+\mathbf{h})-\mathbf{r}(\mathbf{t})}{h}$ for small values of h
   - its *velocity vector* is $\mathbf{v}(t) = \mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(\mathbf{t}+\mathbf{h})-\mathbf{r}(\mathbf{t})}{h}$. The velocity vector is the tangent vector and it points in the direction of the tangent line
   - its *speed* is the magnitude of the velocity: $|\mathbf{v}(t)| = |\mathbf{r}'(t)| = \frac{ds}{dt}$ = the rate of change of distance with respect to time
   - *acceleration vector* is $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$

3. vector integrals help in finding the above values as well, if the acceleration is known (and initial velocity and position):
   - its *velocity vector* is $\mathbf{v}(t) = \mathbf{v}(t_0) + \int_{t_0}^{t} \mathbf{a}(u)du$
   - its *position vector* is $\mathbf{r}(t) = \mathbf{r}(t_0) + \int_{t_0}^{t} \mathbf{v}(u)du$

4. the acceleration of a particle can also be found using Newton's Second Law of Motion: $\mathbf{F}(t) = m\mathbf{a}(t)$